Mathematics A : Examples

< Calculus >

1. Consider the following function of two variables

$$f(x,y) = x^3 - y^3 - 3x + 3y$$.

For this function, the following equations are satisfied at points (a,b),

$$\frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = 0 .$$

(1) Find $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y^2}$.

(2) Find points (a,b) and then calculate $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y^2}$ at (a,b).

(3) Write second-order approximation of $f(x,y)$ at (a,b) by using Taylor expansion.

(4) Find relative maximum, relative minimum and saddle points using the equation obtained in question (3).

2. Consider a function of two variables, $f(x,y)$, for which the following equations are satisfied at points (a,b),

$$\frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = 0 .$$

(1) Find the 2×2 matrix $\boldsymbol{H}$ satisfying the following equation,

$$f(x,y) = f(a,b) + \frac{1}{2}\begin{pmatrix} x-a & y-b \end{pmatrix} \boldsymbol{H} \begin{pmatrix} x-a \\ y-b \end{pmatrix},$$

which is derived by using second-order approximation of Taylor expansion at (a,b).

(2) Consider $\boldsymbol{H}$ in problem (1) has two different eigenvalues λ_1 and λ_2, and corresponding eigenvectors $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$. Show the following equation,

$$\begin{pmatrix} \boldsymbol{p}_1 & \boldsymbol{p}_2 \end{pmatrix}^{-1} = \begin{pmatrix} \boldsymbol{p}_1 & \boldsymbol{p}_2 \end{pmatrix}^{T} .$$

Here the magnitude of the eigenvectors is unit.

(3) By usign the relationship given in problem (2), show the following equation,

$$f(x,y) = f(a,b) + \frac{1}{2}\begin{pmatrix} X & Y \end{pmatrix}\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix} .$$

And then find the condition on λ_1 and λ_2 when the point (a,b) becomes a relative maximum point.

(4) Find relative maximum points of the following function,

$$f(x,y) = x^4 - 2x^2y^2 - y^4 - 2x^2 + 2y^2 .$$

< Linear algebra >

1. The 2×2 matrix $\boldsymbol{A}$ is given by

$$\boldsymbol{A} = \begin{pmatrix} 4 & 2 \\ -1 & 1 \end{pmatrix} .$$

(1) Find the eigenvalues and corresponding eigenvectors of $\boldsymbol{A}$.

(2) Find the 2×2 diagonal matrix $\boldsymbol{D}$ and 2×2 matrix $\boldsymbol{P}$, which satisfy

$\boldsymbol{A} = \boldsymbol{PDP}^{-1}$.

(3) Find $\boldsymbol{A}^n$.

2. The 3×3 matrix $\boldsymbol{A}$ is given by

$$\boldsymbol{A} = \begin{pmatrix} 0 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{pmatrix} .$$

(1) Find the eigenvalues and corresponding eigenvectors of $\boldsymbol{A}$.

(2) Find the 3×3 diagonal matrix $\boldsymbol{D}$ and 3×3 matrix $\boldsymbol{P}$ which satisfy $\boldsymbol{A} = \boldsymbol{PDP}^{-1}$.

(3) Find $\boldsymbol{A}^n$.

3. The 3×3 matrix $\boldsymbol{A}$ is given by

$$\boldsymbol{A} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 2 & -1 & -3 \end{pmatrix}.$$

(1) Find three eigenvalues of matrix $\boldsymbol{A}$, $\lambda_1, \lambda_2, \lambda_3$ $(\lambda_1 < \lambda_2 < \lambda_3)$ and corresponding eigenvectors $\boldsymbol{p}_1$, $\boldsymbol{p}_2$, $\boldsymbol{p}_3$, respectively. Here $|\boldsymbol{p}_1| = |\boldsymbol{p}|_2 = |\boldsymbol{p}_3| = 1$

（2）Find matrices satisfying $\boldsymbol{A} = \boldsymbol{PDP}^{-1}$ by using results of problem (1). Here $\boldsymbol{D}$ is 3×3 diagonal matrix, and $\boldsymbol{P}$ is 3×3 diagonal matrix.

（3）Find $\boldsymbol{A}^n$.

< Vector analysis >

1. In the Cartesian coordinate system (x, y, z), the vector field is given by

$$\boldsymbol{A} = (x - y + y^2 + z^2, y - z + x^2 + z^2, -x + z + x^2 + y^2 + 2y(-x + z)),$$

and the area of region D is defined by

$$D = \{(x, y, z) \mid y^2 + z^2 \le 1\} \quad .$$

(1)Calculate $\nabla \cdot \boldsymbol{A}$ and $\nabla \times \boldsymbol{A}$.

(2) Draw the region D.

(3) Let S be the plane of $x + z = 1$ in the region D. Obtain both the area and shape of S.

(4) Calculate $\int_S \nabla \times \boldsymbol{A} \cdot \boldsymbol{n} dS$, where S is the reigon defined in question(3) and $\boldsymbol{n}$ is the unit normal vector of S with a non-negative z component.

Evaluate $\int_C \boldsymbol{A} \cdot d\boldsymbol{r}$. Here C is a loop around the region S obtained in question (3).

2. In the Cartesian coordinate system (x, y, z), the area of region D and the plane S are defined by

Region: $D = \{(x, y, z) \mid x^2 + y^2 + z^2 \le 1\}$, Plane: $S = \{(x, y, z) \mid x + z = 1\}$.

Let S_D be the plane S in the region D. The vector field $\boldsymbol{A}$ is given by

$$\boldsymbol{A} = (-y, x - z, y) \quad .$$

(1) Calculate $\nabla \times \boldsymbol{A}$.

(2) Calculate the distance between the plane S and origin.

(3) Obtain both the shape and area of S_D.

(4) Calculate $\int_{S_D} \nabla \times \boldsymbol{A} \cdot \boldsymbol{n} dS$, where $\boldsymbol{n}$ is the unit normal vector of S with a non-negative z component. And then, obtain $\int_C \boldsymbol{A} \cdot d\boldsymbol{r}$ by using Stokes theory. Here C is a loop around the region S_D.

(5) Calculate directly $\int_C \boldsymbol{A} \cdot d\boldsymbol{r}$. Here C is a loop defined in problem (4).

3. In the Cartesian coordinate system (x, y, z), the surface S is defined by

$$S = \{(x, y, z) \mid x^2 + y^2 + z^2 = 4, x \geq 0, y \geq 0, z \geq 1\}.$$

A point $p\,(x, y, z)$ is locating on the S, and its position vector is $\boldsymbol{r}$. And $u = x, v = y$.

(1) Express z by using u and v, and then find two vectors, $\dfrac{\partial \boldsymbol{r}}{\partial u}$ and $\dfrac{\partial \boldsymbol{r}}{\partial v}$ at point p.

(2) Using the results of problem (1), find normal unit vector, $\boldsymbol{n}$ on the surface S. Here the z-component of $\boldsymbol{n}$ is not negative.

(3) Surface area of S is calculated by the following equation,

$$\int_S dS = \int_{S^*} f(u, v)\, du\, dv.$$

Draw the integration area, S^*. And find $f(u, v)$ to calculate the above integration.