<u>Mathematics B</u>

1. Let $u(x,t)$ be the real function satisfying the following partial differential equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \ ,$$

with the boundary conditions

$$u(x,0) = f(x) \ ,$$

and

$$u(\infty,t) = u(-\infty,t) = 0 \ ,$$

where $f(x)$ is another real function absolutely integrable over an arbitrary interval. Solve the following problems.

(1) Derive that $u(x,t)$ can be represented as

$$u(x,t) = \int_0^\infty \{A(k)\cos(kx) + B(k)\sin(kx)\}\exp\{-k^2 t\}\,dk \ .$$

(2) Show $A(k)$ and $B(k)$ in problem (1), using $f(x)$. Use the Fourier integral theorem

$$g(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(y)e^{-i\omega(y-x)}dyd\omega \ ,$$

and Euler's formula

$$e^{i\theta} = \cos\theta + i\sin\theta \ ,$$

if necessary.

(3) Find $u(x,t)$ when

$$f(x) = \begin{cases} 1 & (\,|x| \le d) \\ 0 & (\,|x| > d) \end{cases} \ .$$

(4) For $u(x,t)$ of problem (3), find the value of $\lim_{t\to\infty} u(x,t)$.

2. The Fourier series of a function $f(x)$ with period $2L$ is expressed by

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

Solve the following problems.

(1) Prove that the Fourier coefficients can be written as

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx,$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx$$

(2) Using the result of problem (1), obtain the Fourier series of the periodic function $g(x)$ with period $2L$ $(g(x + 2L) = g(x))$:

$$g(x) = x^2, (-L \leq x < L)$$

(3) Using the result of problem (2), find the value of the following infinite series:

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}.$$

3. The Laplace transform of a function $f(x)$ is defined by

$$\mathcal{L}(f(x)) = F(s) = \int_0^\infty f(x)e^{-sx}dx \quad .$$

Solve the following problems.

(1) Obtain the Laplace transforms of the following functions together with their regions of convergence. Note a is a constant.

a) $\sin ax$

b) $\sinh ax$

(2) The convolution of $f(x)$ and $g(x)$, denoted as $f * g$, is given as

$$f * g = \int_0^x f(x-y)g(y)dy \,.$$

Derive

$$\mathcal{L}(f * g) = \mathcal{L}(f)\mathcal{L}(g) \,.$$

(3) Obtain $f(x)$ that satisfies $F(s) = \frac{1}{s^4-1}$.

4. The Fourier transform of a function $f(x)$ is defined by

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt,$$

and the inverse Fourier transform of a function $F(\omega)$ is defined by

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega t}d\omega.$$

These two functions satisfy the following equation known as Parseval's theorem

$$\int_{-\infty}^{\infty} \{f(t)\}^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \{F(\omega)\}^2 d\omega.$$

Solve the following problems.

(1) Obtain the Fourier transform of function $g(t)$ given as

$$g(t) = e^{-|t|}.$$

(2) Using the result of problem (1), obtain the Fourier transform of function $h(t)$ given as

$$h(t) = \frac{1}{1+t^2}.$$

(3) Using the result of problem (1) and Parseval's theorem, calculate

$$\int_{-\infty}^{\infty} \frac{1}{(1+t^2)^2}dt.$$